CALCULATION OF THE TEMPERATURES OF COMPLEX BODIES IN REGULAR THERMAL MODES

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A method is proposed for the semiempirical computation of the temperatures at several points or the mean-bulk temperature of complex systems described by linear heat-conduction equations. The temperature of the surrounding medium depends on the time.

Application of a simple regular mode, as well as of regular modes of the second and third kind, is well known [1, 2]. An attempt is made below to use a broad class of regular thermal modes for the semiempirical computation of the temperature at several points, or of the mean-bulk temperature, of complex bodies on the basis of a single approach to the calculation of the temperature field. The method developed is convenient in those cases when, for technical reasons, it is difficult to obtain a record of the abovementioned temperatures that is continuous in time.

1. Let there be a system consisting of a finite number of bodies in thermal contact, and let the "response," which is the temperature field of the system, be linear in the "input pulse," which is the time dependence of the temperature of the surrounding medium. This latter is valid under the following conditions:

a) the temperature field of the system is described by linear equations of heat conduction without sources and is continuous on the inner boundaries of the system together with the normal heat-flux component (the thermophysical characteristics of the system can hence depend piecewise-continuously on the space coordinates in an arbitrary manner);

b) heat exchange with the external medium is governed by the heat-transfer coefficient which is piecewise-continuous on the outer boundaries of the system;

c) the temperature of the system at the initial instant is zero.

Then, as is known [3], the solution of the problem of the temperature field can be represented as the Duhamel integral

$$t(\tau, \bar{r}) - t_0 = \int_0^\tau \{\varphi(\lambda) - t_0\} \frac{\partial}{\partial \tau} f(\tau - \lambda, \bar{r}) d\lambda. \tag{1.1}$$

The validity of (1.1), which is a particular case of a convolution integral, is the general property of linear systems of diverse nature. Hence, as is done for linear systems, a numerical dependence $f(\tau, \bar{r})$ can be obtained experimentally at several points of the system and then a considerable volume of integro-differential numerical operations can be performed to evaluate $t(\tau, \bar{r})$ at these points for a given function $\varphi(\tau)$ in conformity with (1.1).

However, it is interesting to obtain analytically an explicit dependence of the temperature field on the time and certain parameters of the boundary temperature, especially in those cases when it is difficult, with sufficient accuracy, to find the time derivative of the function $f(\tau, r)$ from experiment or to make a continuous record of this function at several points of the system. The analytical solution of this problem turns out to be realizable in regular thermal modes.

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2. According to the fundamental hypothesis of the regular thermal mode [1, 2], the function $f(\tau, \bar{r})$ can be represented as

$$f(\tau, \bar{r})=1-\sum_{n=0}^{\infty} u_n(\bar{r}) \exp[-m_n\tau], \tag{2.1}$$

where m_n is a sequence of positive numbers, as we specify, in increasing order, and $u_n(\bar{r})$ is a function of the space coordinates, determined uniquely by the thermophysical characteristics of the system and the heat-transfer coefficient. Here evidently

$$\sum_{n=0}^{\infty} u_n(\bar{r}) = 1. \tag{2.2}$$

Applying the Duhamel integral term by term to (2.1), we obtain

$$t(\tau, \bar{r}) - t_0 = \sum_{n=0}^{\infty} m_n u_n(\bar{r}) \exp[-m_n\tau] \int_0^{\tau} \{\varphi(\lambda) - t_0\} \exp[m_n\lambda] d\lambda. \tag{2.3}$$

The validity of term-by-term application of the Duhamel integral is proved in the Appendix. The integrals in the right side of (2.3) are simple in structure and easily evaluated analytically for some important particular cases of the time dependence of the temperature of the ambient medium. The result of calculating the temperature field in conformity with (2.3), taking account of (2.2), has the form of a sum of several terms with a simple time dependence and some quantity S which is the sum of an infinite series consisting of terms decreasing exponentially with time. The quantity S depends in a complex manner on the time, but its absolute and relative contribution to the temperature field tends to zero as $\tau \to \infty$. Passing to the regular thermal mode, i. e., neglecting S, after simple but tedious calculations we obtain the results presented in Table 1. Here $\gamma_0, \gamma_1, \ldots$ and ω are parameters of the time dependence of the temperature of the ambient medium; $k_s(\bar{r})$, $\lambda_1(\bar{r})$, $\lambda_2(\bar{r}, \omega)$, $F_1(\bar{r}, \omega^2)$, $F_2(\bar{r}, \omega^2)$, $\Psi_1(\bar{r}, \omega^2)$, $\Psi_2(\bar{r}, \omega^2)$ are functions of the variables indicated in parentheses. The dependence of these functions on $\bar{r}$, ω, ω^2 is defined uniquely by the quantities m_n and $u_n(\bar{r})$ from (2.1), i. e., the thermophysical characteristics of the system and the heat-transfer coefficients. Thus, for example,

$$k_s(\bar{r}) = \sum_{n=0}^{\infty} m_n^{-s} u_n(\bar{r}),$$

and the remaining functions are no less complex and their analytical calculation is not possible in the general case. For the sequel it is essential only that these functions be independent of the time and γ_0, $\gamma_1, \ldots$.

Let us now note the following:

a) the broad class $\varphi(\tau)$ is approximated on considerable time segments by a linear combination of polynomials, exponentials, trigonometric functions, etc.

b) from the linearity of (1.1) in $\{\varphi(\tau) - t_0\}$ there follows

$$t(\tau, \bar{r}) = \sum_{i=0}^{i=\nu} a_i t_i(\tau, \bar{r}) \quad \text{for} \quad \varphi(\tau) - t_0 = \sum_{i=0}^{i=\nu} a_i \{\varphi_i(\tau) - t_0\},$$

where a_i are constants, $t_i(\tau, \bar{r})$ is the temperature field of the system for $\varphi(\tau) \equiv \varphi_i(\tau)$.

Hence, the temperature field of a system in the regular thermal mode is represented, for a broad class of functions $\varphi(\tau)$, as a linear combination of the results presented in Table 1, and their like.

Therefore, for arbitrary linear systems and a broad class of time dependences of the temperature of the ambient medium, it proves possible to obtain an explicit dependence of the temperature field on the time, the parameters $\gamma_0, \gamma_1, \ldots$, and partially on ω in regular thermal modes. For different systems only m_0 and the functions k_s, $\lambda_{1,2}$, $F_{1,2}$ etc., will be different.

It is easy to see that, on averaging the temperature field in the regular thermal mode over an arbitrary volume of the system, the form of the dependence on the time and the parameters will not change, but the functions dependent on $\bar{r}$ are replaced in an appropriate way by the values averaged over the volume.

As has been mentioned above, the values of the temperature field indicated in Table 1 differ from the true values by the quantity S. Thus,

TABLE 1. Formulas for the Calculation of the Temperature Fields of the System $t(\tau, \bar{r})$

No.	$\varphi(\tau)$	$t(\tau, \bar{r})$ in the regular thermal mode
1	$\varphi(\tau) = \gamma_0 + \gamma_1\tau + \gamma_2\tau^2 + \gamma_3\tau^3$	$t(\tau, \bar{r}) = \gamma_0 + \{\gamma_1 - 2\gamma_2 k_1(\bar{r}) + 6\gamma_3 k_2(\bar{r})\}\tau + \{\gamma_2 - 3\gamma_3 k_1(\bar{r})\}\tau^2 + \gamma_3\tau^3 + \{2\gamma_2 k_2(\bar{r}) - \gamma_1 k_1(\bar{r}) - 6\gamma_3 k_3(\bar{r})\}$
2	$\varphi(\tau) = \sum_{s=0}^{s=i} \gamma_s\tau^s \equiv P_i(\tau)$	$t(\tau, \bar{r}) = \sum_{s=0}^{s=i} (-1)^s k_s(\bar{r}) \frac{d^s}{d\tau^s} P_i(\tau)$
3	$\varphi(\tau) = \gamma_0 + \gamma_1 \exp[-\omega\tau]$	$t(\tau, \bar{r}) = \gamma_0 + \gamma_1\lambda_1(\bar{r}) \frac{m_0}{m_0 - \omega} \{\exp[-\omega\tau] - \exp[-m_0\tau]\} + \gamma_1\lambda_2(\bar{r}, \omega) \exp[-\omega\tau]$
4	$\varphi(\tau) = \gamma_0 + \gamma_1\tau \cos[\omega\tau + \theta]$	$t(\tau, \bar{r}) = \gamma_0 + \gamma_1 F_1(\bar{r}, \omega^2)\tau \cos\{\omega\tau + \theta + \Psi_1(\bar{r}, \omega^2)\} + \gamma_1 F_2(\bar{r}, \omega^2) \cos\{\omega\tau + \theta + \Psi_2(\bar{r}, \omega^2)\}$
5	$\varphi(\tau) = \gamma_0 + \gamma_1 \cos[\omega\tau + \theta]$	$t(\tau, \bar{r}) = \gamma_0 + \gamma_1 F_1(\bar{r}, \omega^2) \cos\{\omega\tau + \theta + \Psi_1(\bar{r}, \omega^2)\}$

$$S = \sum_{n=0}^{\infty} u_n(\bar{r})\{t_0 - \gamma_0 + \gamma_1 m_n^{-1} - 2\gamma_2 m_n^{-2} + 6\gamma_3 m_n^{-3}\} \exp[-m_n\tau].$$

in the first example in Table 1.

It is interesting to note, however, that in order to take account of the influence of the initial temperature t_0, one or several of the first terms of the series can be retained, to obtain, if it can thus be expressed, a "higher-order" regular mode.

3. The results presented permit several deductions.

A general principle is expressed in [4] according to which, in the regular thermal mode,

$$\frac{d\langle t\rangle}{d\tau} = c\{\langle t\rangle - \varphi(\tau)\}, \tag{3.1}$$

where $\langle t\rangle$ is the mean-bulk temperature and c is a constant.

For example, let $\varphi(\tau) = \gamma_0 + \gamma_1\tau + \gamma_2\tau^2 + \gamma_3\tau^3$. Comparing the solution of (3.1) with the result presented in Table 1, it is easy to see that $c = -\langle k_1\rangle^{-1}$ and that they agree only in the particular case

$$\langle k_3\rangle = \langle k_1\rangle^3, \quad \langle k_2\rangle = \langle k_1\rangle^2,$$

where $\langle k_s\rangle$ is the mean-bulk value of the function $k_s(\bar{r})$. They differ in the remaining cases by the quantity

$$6\gamma_3\{\langle k_2\rangle - \langle k_1\rangle^2\}\tau + 2\gamma_2\{\langle k_2\rangle - \langle k_1\rangle^2\} - 6\gamma_3\{\langle k_3\rangle - \langle k_1\rangle^3\},$$

the relative contribution of which to the temperature field tends to zero as $\tau \to \infty$, while the absolute contribution tends to infinity. In the other examples in Table 1, c is a function of ω. Hence, the application of the Duhamel integral permits the meaning of c to be revealed and allows the principle expounded above to be refined in the sense that it correctly yields the function $\langle t\rangle$ with a small relative, but no absolute, error.

Of special interest is the case of an exponential time dependence of the temperature of the surrounding medium. Passing to the limit $\omega \to m_0$ in example 3 in Table 1, we obtain for $\omega = m_0$

$$t(\tau, \bar{r}) = \gamma_0 + \gamma_1\lambda_1(\bar{r}) m_0\tau \exp[-m_0\tau], \tag{3.2}$$

i. e., an original resonance phenomenon sets in when the system temperature varies according to the same exponential law but the preexponential factor, i. e., the amplitude of the exponential, tends to infinity as $\tau \to \infty$. For $\omega \gg m_0$ the dependence on ω is also written explicitly, the heating being limited by the system inertia

$$t(\tau, \bar{r}) = \gamma_0 + \gamma_1\lambda_1(\bar{r}) m_0\omega^{-1} \exp[-m_0\tau]. \tag{3.3}$$

4. Practical application of the results obtained can be achieved by an experimental determination of the values of the functions k_s, $\lambda_{1,2}$, $F_{1,2}$, etc., at one or several points of the system, or of their mean-bulk

values. For example, let the temperature of the surrounding medium depend exponentially on the time. According to example 3 in Table 1, it is sufficient to determine experimentally, in a regular thermal mode, the value of the temperature field $t(\tau, \bar{r}_0)$ at a fixed point $\bar{r}_0$ of the given system for a fixed ω at three moments of time, in order to then compute m_0 and the values of the functions $\lambda_1(\bar{r}_0)$, $\lambda_2(\bar{r}_0, \omega)$ at this point. Having determined these functions, the value of the temperature field in a regular thermal mode can be computed by the formula of example 3 at the point $\bar{r}_0$ at arbitrary times and for any γ_0, γ_1 but fixed ω. The computation of the mean-bulk temperature of an arbitrary part of the system is accomplished by the same method of replacing the functions $\lambda_1(\bar{r})$, $\lambda_2(\bar{r}, \omega)$ by their mean-bulk values. The mean-bulk values of these functions is achieved by the method of three measurements of the mean-bulk temperature which is realizable, for example, by a calorimetric study of the part of the system under investigation. According to (3.2) and (3.3), for $\omega = m_0$ and $\omega \gg m_0$ two measurements are sufficient. The quantity m_0 is also determined by methods of the simple regular mode [1, 2].

In other cases, the dependences of the temperature of the surrounding medium on the time of operation are perfectly analogous and make it possible to bypass the problem of determining the thermophysical characteristics of complex systems and the distribution of the heat-transfer coefficient.

APPENDIX

The correctness of the operation of term-by-term application of the Duhamel integral should be analyzed separately. Let us prove the following assertion. For $\tau \in [0, \tau_0)$ let

1) the series comprised of functions $f_n(\tau)$ converge uniformly to the function $f(\tau)$:

$$f(\tau) = \sum_{n=0}^{\infty} f_n(\tau),$$

where $f_n(\tau)$ is a function of τ and perhaps still other parameters which are not written explicitly here;

2) the functions $f_n(\tau)$, $f(\tau)$, $\varphi(\tau)$ be continuously differentiable. Then

$$\int_0^{\tau} \frac{\partial f(\tau-\lambda)}{\partial \tau} \varphi(\lambda)\, d\lambda = \sum_{n=0}^{\infty} \int_0^{\tau} \frac{\partial f_n(\tau-\lambda)}{\partial \tau} \varphi(\lambda)\, d\lambda.$$

Proof. According to the second condition, all the integrals in (1) exist and can be integrated by parts, whilst the derivative $\partial\varphi(\tau-\lambda)/\partial\lambda$ is bounded in the segment $[0, \tau]$ and, therefore [5], the series $\sum_{n=0}^{\infty} f_n(\lambda)(\partial\varphi(\tau-\lambda)/\partial\lambda)$ converges uniformly, by virtue of the first condition, and admits of term-by-term integration. Hence, we have

$$\int_0^{\tau} \frac{\partial f(\tau-\lambda)}{\partial \tau} \varphi(\lambda)\, d\lambda = \int_0^{\tau} \frac{\partial f(\lambda)}{\partial \lambda} \varphi(\tau-\lambda)\, d\lambda = f(\tau)\varphi(0) - f(0)\varphi(\tau)$$

$$- \int_0^{\tau} f(\lambda) \frac{\partial \varphi(\tau-\lambda)}{\partial \lambda}\, d\lambda = \sum_{n=0}^{\infty} \Big\{ f_n(\tau)\varphi(0) - f_n(0)\varphi(\tau)$$

$$- \int_0^{\tau} f_n(\lambda) \frac{\partial \varphi(\tau-\lambda)}{\partial \lambda}\, d\lambda \Big\} = \sum_{n=0}^{\infty} \int_0^{\tau} \frac{\partial f_n(\tau-\lambda)}{\partial \tau} \varphi(\lambda)\, d\lambda,$$

QED. The assertion is extended directly to the case of piecewise-continuous derivatives.

The continuous differentiability of the members of the series (2.1) is evident. From physical considerations it is also clear that the rate of temperature rise within the system $f(\tau, \bar{r})$ (for constant temperature of the surrounding medium) is piecewise continuous. The requirements of piecewise-continuity of the rate of temperature rise of the temperature of the ambient medium $\varphi(\tau)$ is always satisfied. The convergence of the series (2.1), by the Abel criterion [5], implies its uniform convergence, since the series $\sum_{n=0}^{\infty} u_n(\bar{r})$ (independent of τ) converges uniformly, and $\exp[-m_n\tau]$ form, for fixed τ, monotonic sequences bounded in a set.

Therefore, in heat-conduction problems the series (2.1) and $\varphi(\tau)$ completely satisfy the conditions of the assertion and admit of term-by-term application of the Duhamel integral.

NOTATION

$t(\tau, \bar{r})$, $t_i(\tau, \bar{r})$	are temperature distributions within the body;
$\varphi(\tau)$, $\varphi_i(\tau)$	are time dependences of the temperature of the surrounding medium;
t_0	is the initial body temperature;
$f(\tau, \bar{r})$	is the temperature field for $t_0 = 0$, $\varphi(\tau) = 1$;
$u_n(\bar{r})$	are functions of the coordinates of points of the body;
γ_s, ω, ϑ	are parameters of the function $\varphi(\tau)$;
$P_i(\tau)$	is a polynomial of the i-th power in τ;
$k_s(\bar{r})$, $\lambda_1(\bar{r})$, $\lambda_2(\bar{r}, \omega)$, $F_{1,2}(\bar{r}, \omega)$, $\Psi_{1,2}(\bar{r}, \omega)$	are functions characterizing the dependence of the temperature field on the coordinates of points of the body in regular modes;
a_i, c	are constants;
S	is the sum of an infinite series which is the deviation of the true behavior of the temperature from the simple dependence calculated in the regular thermal mode;
$\bar{r}$	is the radius-vector.

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